Implementation of the DKSS Algorithm for Multiplication of Large Numbers

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Introduction

- In 2008, De, Kurur, Saha & Saptharishi (DKSS) published a paper on how to multiply large numbers based on ideas of Fürer's algorithm.
- Their procedure was implemented and compared to Schönhage-Strassen multiplication to see how it performs in practice.

- *But first, some context...*

Representation of Large Numbers

- On 64-bit machines a *word* can hold non-negative values $< W = 2^{64}$.
- A large number $0 \le a < W^n$ is represented as array of n words: $(a_0, a_1, \ldots, a_{n-1})$.
- Each word a_i is a "digit" of a in base W.
- Ordinary (grade-school) multiplication of $a \cdot b$: multiply each a_i with each b_j. Run-time is $O(n^2)$. Function name `OMUL`.

- *Can we do better?*

Multiplication: Karatsuba

- (Karatsuba 1960): cut numbers a and b in half. With the help of some linear time operations, only 3 half-sized multiplications are needed:

$$a = a_0 + a_1 W^n, \qquad b = b_0 + b_1 W^n$$
$$P_0 = a_0 b_0, \qquad P_1 = (a_0 - a_1)(b_0 - b_1), \qquad P_2 = a_1 b_1$$
$$ab = P_0(1 + W^n) - P_1 W^n + P_2(W^n + W^{2n})$$

- When done recursively run-time is $O(n^{\log_2 3}) \approx O(n^{1.58})$. Function name `KMUL`.

Multiplication: Toom-Cook

- (Toom 1963, Cook 1966): cut numbers in $k \geq 2$ pieces and perform only $2k-1$ "small" multiplications plus some linear time operations.
- Run-time is $O(n^{\log_k(2k-1)})$. For $k = 3, 4, 5$ this is $\approx O(n^{1.46})$, $O(n^{1.40})$, $O(n^{1.37})$. Function name for $k = 3$ is `T3MUL`.
- Problem: the number of linear time operations grows quickly with k.

Multiplication: FFT-Methods

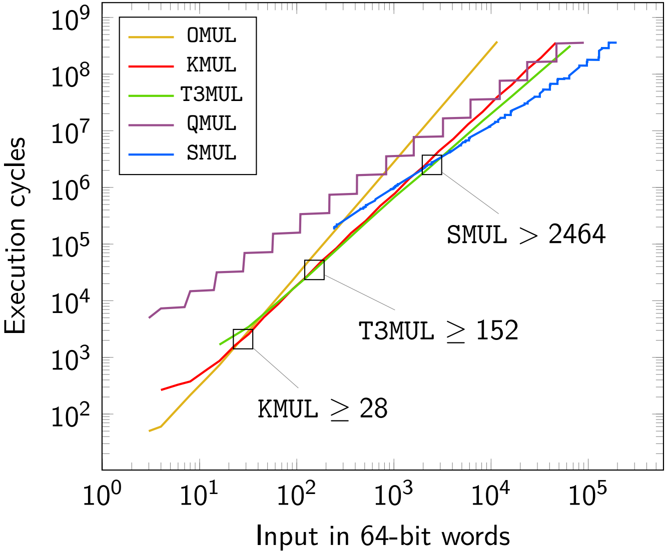
- (Strassen 1968): Cut numbers a and b in $n/2$ pieces each and interpret pieces as coefficients of polynomials over $R[x]$, R ring.
- Evaluate polynomials at n points, multiply the sample values and interpolate to obtain product. Propagate carries.
- If ω is primitive n-th root of unity in R, evaluation and interpolation can be done on ω^k, $0 \leq k < n$. We can use the fast Fourier transform (FFT) with $O(n \cdot \log n)$ steps. Function name `QMUL`.
- Problem: the larger n becomes, the more precision is needed in coefficient ring R. This limits the length of input numbers.

Multiplication: Schönhage-Strassen

- (Schönhage & Strassen 1971): Use $R = \mathbb{Z}/(2^K + 1)\mathbb{Z}$ and $\omega = 2$ as primitive $2K$-th root of unity for the FFT.
- Multiplications by ω^k are just cyclic shifts, can be done in linear time.
- Run-time is $O(N \cdot \log N \cdot \log\log N)$, coefficient length is $O(\sqrt{N})$. Function name `SMUL`.

- Problem: the order of ω is not very high. Except for $\sqrt{2}$, there are generally no higher order roots of unity, thus FFT length is quite limited.
- Nevertheless, Schönhage-Strassen is the standard for multiplication of large numbers with over $\approx 150\,000$ bits.

Crossover Points Between Algorithms



Multiplication: DKSS

- (De, Kurur, Saha & Saptharishi 2008): Use polynomial quotient ring $R = \mathcal{P}[\alpha]/(\alpha^m + 1)$ with $\mathcal{P} = \mathbb{Z}/p^c\mathbb{Z}$, $p = h \cdot 2M + 1$ prime.
- Select $M = N/\log^2 N$ and $m = \log N$ as powers of 2, $M > m$. Let $\mu = M/m$.
- From a generator of $\mathbb{F}_p^*$ calculate a primitive $2M$-th root of unity $\rho \in \mathcal{P}[\alpha]$ with $\rho^\mu = \alpha$.

- With α as primitive $2m$-th root of unity and modulus $(\alpha^m + 1)$ multiplications by α^k are cyclic shifts: fast!
- ρ is high order root of unity: large FFT length.

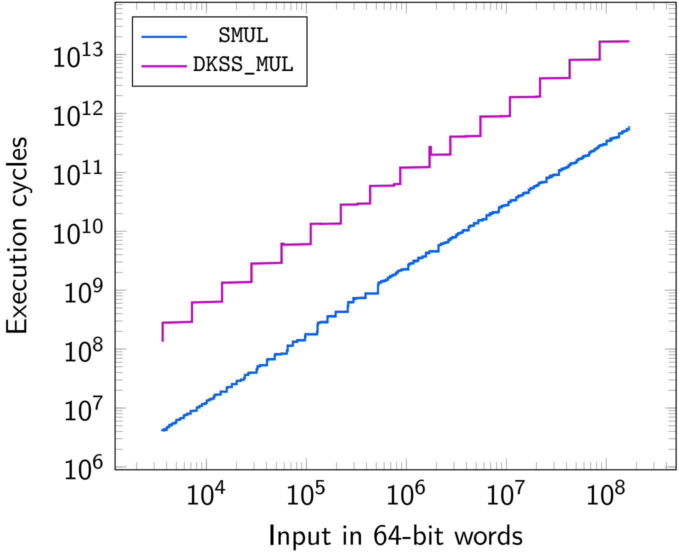
Multiplication: DKSS (continued)

- A length-$2M$ FFT can be calculated like this:
  - $2M = \mu \cdot 2m$.
  - Interpret the coefficients as a matrix with $2m$ rows and μ columns.
  - Do μ many length-$2m$ FFTs (on the columns) with α as root of unity.
  - Perform *bad multiplications* on the coefficients, i.e. multiply them by some ρ^k.
  - Do $2m$ many length-μ FFTs (on the rows) by calling the FFT routine recursively.
- Multiplication in R is reduced to integer multiplication by use of Kronecker-Schönhage substitution.
- Run-time is $O(N \cdot \log N \cdot K^{\log^* N})$ with $K = 16$, coefficient length is $O(\log^2 N)$. Function name `DKSS_MUL`.

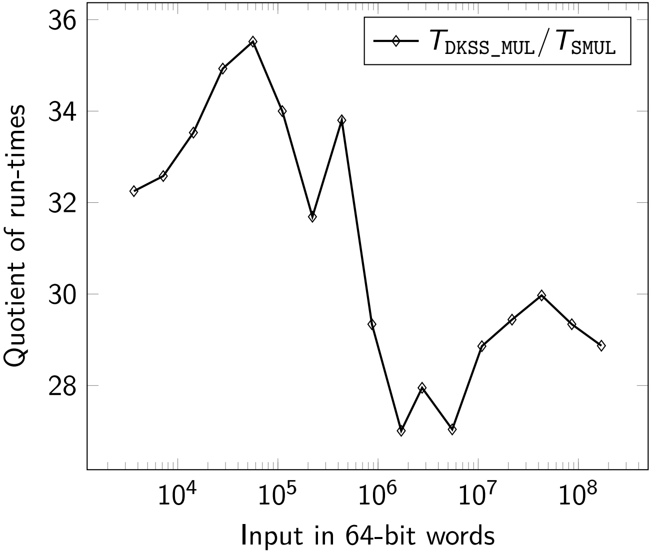
Multiplication: Simplified DKSS

- In genuine DKSS, prime p is searched at run-time. To keep that time low, p must be kept small. So, input numbers are encoded as k-variate polynomials, k constant.
- Since input length is limited by available memory, we can precompute all of the required primes p and generators of $\mathbb{F}_p^*$.
- This allows to use univariate polynomials and simplifies calculation of the root of unity ρ. We can use $c = 1$ and hence $\mathcal{P} = \mathbb{Z}/p\mathbb{Z}$.
- For 64-bit architecture, only 6 primes need to be precomputed.

Comparison of Execution Time



Quotient of Run-times



Results

For the numbers tested (up to 1.27 GB input size, total temporary memory required 26 GB):

- DKSS_MUL is between 27 and 36 times slower than SMUL.
- DKSS_MUL requires $\approx$ 2.3 times the temporary memory than SMUL.
- About 80 % of run-time is spent with *bad multiplications*, i.e. multiplications by ρ^k that are not powers of α.
- Another 9 % are spent for pointwise products.
- Recursion did not take place. Even with the largest inputs, inner multiplications were just 195 words long.
- Cache effects did not slow it down, either.

When Will DKSS Beat Schönhage-Strassen?

- Model `SMUL` run-time:

$$T_\sigma \le \sigma \cdot N \cdot \log N \cdot \log\log N.$$

- Model `DKSS_MUL` run-time:

$$T_\eta \le \eta \cdot N \cdot \log N \cdot K^{\log^* N}, \quad K = 16.$$

- Find fitting constants σ and η from measured run-times.
- Solve $T_\sigma \ge T_\eta$ numerically:

$$N \ge 10^{10^{4796}} \quad !!$$

Future work

Some ideas:

- Exploit the sparseness of the factors in the underlying multiplication. Estimated speed-up: factor 2.
- Use variant of Kronecker-Schönhage substitution (Harvey).
- Parameters p, M and m should be selected with more care. Estimated speed-up: maybe 30 %.
- Modular reduction should be sped up (Montgomery's trick or other). Estimated speed-up: about 22 %.
- Total estimated possible speed-up: factor 3.2, but even then `DKSS_MUL` is at best 8.5 times slower than `SMUL`.

Source Code & Thanks

- Implementation was done in C++ and assembly language under Windows as part of `BIGNUM`, my large integer library.
- Multiplication compares favorably with MPIR (GMP for Windows) and is only 1.3 times slower on average.
- Source code is available from http://www.wrogn.com/bignum and licensed under LGPL.

- Many thanks to Andreas Weber and Michael Clausen.